

The Quest for Rings on Bipolar Scales

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We consider the interval $] -1, 1[$ and intend to endow it with an algebraic structure like a ring. The motivation lies in decision making, where scales that are symmetric w.r.t. 0 are needed in order to represent a kind of symmetry in the behaviour of the decision maker. A former proposal due to Grabisch was based on maximum and minimum. In this paper, we propose to build our structure on t-conorms and t-norms, and we relate this construction to uninorms. We show that the only way to build a group is to use strict t-norms, and that there is no way to build a ring. Lastly, we show that the main result of this paper is connected to the theory of ordered Abelian groups.

Keywords: ordered group, pseudo-addition, pseudo-multiplication, t-conorm, t-norm, uninorm.

1. Introduction

So far, most of the studies of aggregation operators have been conducted on the unit interval $[0, 1]$, being representative for membership degrees for fuzzy sets, uncertainty degrees for probability measures and other non-classical measures, etc. From psychological studies, it is known that human beings handle other kinds of scales. Three kinds of scales have been identified:

- *bounded unipolar scales*: typically $[0, 1]$, suitable for bounded notions such as membership degrees and uncertainty degrees;

- *unipolar scales* (not necessarily bounded): typically \mathbb{R}^+ , suitable e.g. for priority degrees (one can always imagine something of higher priority);
- *bipolar scales* (bounded or unbounded): typically \mathbb{R} , suitable for all paired concepts of natural language, such as attraction/repulsion, good/bad, etc.

In decision making, it has been shown that the use of bipolar scales is of particular interest, since they enable the representation of symmetry phenomena in human behaviour, when one is faced with positive (gain, satisfaction, etc.) or negative (loss, dissatisfaction, etc.) scores or utilities. This has led for example to models in decision under uncertainty such as Cumulative Prospect Theory¹⁵, based on the symmetric Choquet integral.

Recently, Grabisch has investigated symmetric algebraic structures imitating the ring structure of real numbers, where the usual $+$ and \times operations were replaced by suitable extensions of maximum \vee and minimum \wedge ^{5,7}. The aim was to build an ordinal counterpart of numerical bipolar scales, and of the symmetric Choquet integral, in order to make some first steps towards an ordinal version of Cumulative Prospect Theory. He proposed new operations, called symmetric maximum and minimum, and obtained a structure close to a ring. In fact, associativity was problematic, but it was nevertheless possible to define a symmetric Sugeno integral, counterpart of the symmetric Choquet integral⁶.

Symmetric maximum and minimum can be seen as extensions of the usual \vee and \wedge on $[-1, 1]$. Recall that \vee (resp. \wedge) is a t-conorm (resp. t-norm) on $[0, 1]$. The question that now comes to mind is the following: Is it possible to extend any pair consisting of a t-conorm and a t-norm to $[-1, 1]$ or $] -1, 1[$ so that the resulting structure is a ring? Or, at least, is it possible to construct an Abelian group? A positive answer to this question would permit readily to define symmetric fuzzy integrals defined with t-conorms and t-norms¹⁴. However, the results we present here indicate that the quest for rings on bipolar scales seems to be a chimeric task.

2. Background

In this section, we introduce the concepts needed for our constructions. We refer the reader to¹¹ for a comprehensive treatment. It is also the source of our notations.

Definition 1 *A triangular norm (t-norm for short) T is a binary operation on $[0, 1]$ such that for any $x, y, z \in [0, 1]$ the following four axioms are satisfied:*

- (P1) *commutativity:* $T(x, y) = T(y, x)$;
- (P2) *associativity:* $T(x, T(y, z)) = T(T(x, y), z)$;
- (P3) *monotonicity:* $T(x, y) \leq T(x, z)$ whenever $y \leq z$;
- (P4) *neutral element:* $T(1, x) = x$.

Any t-norm satisfies $T(0, x) = 0$. Typical t-norms are the minimum (\wedge), the algebraic product (\cdot), and the Łukasiewicz t-norm defined by $T_L(x, y) := (x + y - 1) \vee 0$.

Definition 2 A triangular conorm (*t-conorm for short*) S is a binary operation on $[0, 1]$ that, for any $x, y, z \in [0, 1]$, satisfies **P1**, **P2**, **P3** and

(P5) *neutral element*: $S(0, x) = x$.

Any t-conorm satisfies $S(1, x) = 1$. Typical t-conorms are the maximum \vee , the probabilistic sum $S_P(x, y) := x + y - xy$, and the Łukasiewicz t-conorm defined by $S_L(x, y) := (x + y) \wedge 1$. T-norms and t-conorms are dual operations in the sense that for any given t-norm T , the binary operation S_T defined by

$$S_T(x, y) = 1 - T(1 - x, 1 - y)$$

is a t-conorm (and similarly when starting from S). Hence, their properties are also dual. The above examples are all dual pairs of t-norms and t-conorms.

A t-norm (or a t-conorm) is said to be *strictly monotone* if $T(x, y) < T(x, z)$ whenever $x > 0$ and $y < z$. A continuous t-norm (resp. t-conorm) is shown to be *Archimedean* if $T(x, x) < x$ (resp. $S(x, x) > x$) for all $x \in]0, 1[$. A strictly monotone and continuous t-norm (resp. t-conorm) is called *strict*. Strict t-norms (resp. t-conorms) are Archimedean. Non-strict continuous Archimedean t-norms (resp. t-conorms) are called *nilpotent*.

Any continuous Archimedean t-conorm S has an additive generator s , i.e. a strictly increasing function $s : [0, 1] \rightarrow [0, +\infty]$, with $s(0) = 0$, such that, for any $x, y \in [0, 1]$:

$$S(x, y) = s^{-1}[s(1) \wedge (s(x) + s(y))]. \quad (1)$$

Similarly, any continuous Archimedean t-norm has an additive generator t that is strictly decreasing and satisfies $t(1) = 0$. Strict t-conorms are characterized by $s(1) = +\infty$, nilpotent t-conorms by a finite value of $s(1)$. Additive generators are determined up to a positive multiplicative constant. If t is an additive generator of a t-norm T , then $s(x) = t(1 - x)$ is an additive generator of its dual t-conorm S_T .

Definition 3¹⁷ A uninorm U is a binary operation on $[0, 1]$ that, for any $x, y, z \in [0, 1]$, satisfies **P1**, **P2**, **P3** and

(P6) *neutral element*: there exists $e \in]0, 1[$ such that $U(e, x) = x$.

It follows that on $[0, e]^2$ a uninorm behaves like a t-norm, while on $[e, 1]^2$ it behaves like a t-conorm. In the remaining parts, monotonicity implies that U is comprised between min and max. Associativity implies that $U(0, 1) \in \{0, 1\}$. Uninorms such that $U(0, 1) = 1$ are called disjunctive, while the others are called conjunctive. By a simple rescaling, one can define from a given uninorm U a t-norm T_U and a t-conorm S_U :

$$T_U(x, y) = \frac{1}{e} U(ex, ey) \quad (2)$$

$$S_U(x, y) = \frac{1}{1 - e} [U(e + (1 - e)x, e + (1 - e)y) - e]. \quad (3)$$

Conversely, from a given t-norm T and t-conorm S , one can (partially) define a uninorm $U_{T,S}$ by:

$$U_{T,S}(x, y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & , \text{ if } (x, y) \in [0, e]^2 \\ e + (1 - e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & , \text{ if } (x, y) \in [e, 1]^2. \end{cases} \quad (4)$$

The remaining parts can be filled in with minimum or maximum, leading to two extremal uninorms with the given underlying operators. The problem of filling in the remaining parts is non-trivial and has been solved recently for the case of a pair of continuous operators³. Minimum and maximum, for instance, can be extended in various ways to an idempotent uninorm².

Proposition 1 *The following statements are equivalent:*

- (i) U is a uninorm with neutral element e , strictly monotone on $]0, 1[^2$, and continuous on $[0, 1]^2 \setminus \{(0, 1), (1, 0)\}$.
- (ii) There exists an additive generator u , i.e. a strictly increasing $[0, 1] \rightarrow [-\infty, \infty]$ mapping u such that $u(e) = 0$ and for any $x, y \in [0, 1]$:

$$U(x, y) = u^{-1}(u(x) + u(y)), \quad (5)$$

where by convention $\infty - \infty = -\infty$ if U is conjunctive, and $+\infty$ if U is disjunctive.

Uninorms characterized by the above proposition are called *representable* uninorms.

Under the above assumption, it turns out that T_U and S_U are strict, and have additive generators t_u and s_u defined by:

$$t_u(x) = -u(ex) \quad (6)$$

$$s_u(x) = u(e + (1 - e)x). \quad (7)$$

Definition 4 *Let S be a t-conorm. The S -difference is defined by:*

$$x \ominus'_S y := \inf\{z \mid S(y, z) \geq x\}. \quad (8)$$

S -differences have been proposed by Weber¹⁶. From a logical point of view, S -differences are extensions of the binary coimplication, the logical dual of implication (in the sense of de Morgan)¹. S -differences are dual to residual implicators of t-norms.

If S has an additive generator s , then it is easy to show that for any $x, y \in [0, 1]$:

$$x \ominus'_S y = s^{-1}(0 \vee (s(x) - s(y))). \quad (9)$$

If $S = \max$, then this difference operator becomes:

$$x \ominus'_\vee y = \begin{cases} x & , \text{ if } x > y \\ 0 & , \text{ else.} \end{cases} \quad (10)$$

In lattice theory, $x \ominus'_\vee y$ corresponds to the dual of the pseudo-complement of y relative to x (see e.g.⁸).

3. Symmetric pseudo-additions and pseudo-multiplications

Following our motivation given in the introduction, we try to define in this section what we mean by symmetric pseudo-addition and pseudo-multiplication on $[-1, 1]$. Let us first consider the pseudo-addition. Our aim is to endow $[-1, 1]$ with a binary operation \oplus , extending a given t-conorm S , so that we get a commutative group, with neutral element 0.

A first basic fact is that we should consider the open interval $] -1, 1[$, instead of the closed one if we want to get a group structure (and hence a ring). Indeed, associativity implies that for any $a \in [0, 1[$, we should have $a \oplus (1 \oplus -1) = (a \oplus 1) \oplus (-1)$. Since on $[0, 1]$, \oplus coincides with S , the above equality becomes $a \oplus (1 \oplus -1) = 1 \oplus (-1)$, since 1 is an absorbing element. This equality cannot be satisfied for all $a \in [0, 1[$, unless $1 \oplus (-1)$ is again an absorbing element; in particular we cannot have $1 \oplus (-1) = 0$ as required by the group structure. Hence, we exclude -1 and 1 to get a group, but we can still consider that \oplus is defined on $[-1, 1]^2$, imposing that $1 \oplus (-1)$ is equal to 1 or -1 . As it will be seen in Section 5, the case of the closed interval corresponds to *extended* groups.

We propose the following definition.

Definition 5 *Given a t-conorm S , we define a symmetric pseudo-addition \oplus as a binary operation on $[-1, 1]$:*

- (R1) *For $x, y \geq 0$: $x \oplus y = S(x, y)$.*
- (R2) *For $x, y \leq 0$: $x \oplus y = -S(-x, -y)$.*
- (R3) *For $x \in [0, 1[$, $y \in] -1, 0]$: $x \oplus y = x \ominus_S (-y)$, where \ominus_S is a symmetrized version of the S -difference:*

$$x \ominus_S y = \begin{cases} \inf\{z \mid S(y, z) \geq x\} & , \text{ if } x \geq y \\ -\inf\{z \mid S(x, z) \geq y\} & , \text{ if } x \leq y \\ 0 & , \text{ if } x = y \end{cases}$$

for $x, y \geq 0$. Moreover, $1 \oplus (-1) = 1$ or -1 .

- (R4) *For $x \leq 0, y \geq 0$: just reverse x and y .*

We now give justifications for our definition.

- On $[0, 1]^2$, \oplus has to be commutative, associative, have neutral element 0 and absorbing element 1. Adding monotonicity (which may be questionable), we have exactly the axioms of a t-conorm. This justifies **R1**.
- Since a group structure is desired, one should have a symmetric element $-x$ for each element $x \in] -1, 1[$, i.e. $x \oplus (-x) = 0$. Using associativity and commutativity, this leads to $0 = (x \oplus (-x)) \oplus (y \oplus (-y)) = (x \oplus y) \oplus ((-x) \oplus (-y))$, which implies $(-x) \oplus (-y) = -(x \oplus y)$, which in turn implies **R2**.

- When $x \geq 0$ and $y \leq 0$, we define $x \oplus y := x \ominus_S (-y)$. Although this definition might seem to be somewhat arbitrary, the following considerations justify this choice. For a strict t-conorm S and $x \geq |y|$, a comparison of formulas (9) and (1) clearly shows that they complement each other. For the case of maximum, we refer to Section 4.3. This justifies the first line of **R3**.

Notice also that in the case of a representable uninorm U with neutral element $e \in]0, 1[$, and additive generator u , we always have $U(x, N(x)) = e$ for $x \in]0, 1[$, where $N(x) := u^{-1}(-u(x))$ is a strong negation. After some calculation and appropriate rescaling (i.e. the set $[e, 1] \times [0, e]$ is linearly mapped to $[0, 1] \times [-1, 0]$, whence U becomes \oplus and $N(y)$ becomes $-y$), one can come up with the formula

$$x \oplus y = x \ominus'_S (-y),$$

for $(x, y) \in [0, 1] \times [-1, 0]$, $x \geq -y$. Compare this also with Section 4.1, especially Proposition 2. The second line stems naturally from the requirement $-(x \oplus y) = (-x) \oplus (-y)$, while the third line is due to $x \oplus (-x) = 0$.

- **R4** originates from commutativity.

The case of pseudo-multiplication is less problematic. We want a commutative and associative operation having 1 as neutral element and 0 as absorbing element, and with the same rule of sign as the product, i.e.

$$x \odot y = \text{sign}(x \cdot y) T(|x|, |y|),$$

where the sign function is defined as usual by

$$\text{sign}(x) := \begin{cases} 1 & , \text{ if } x > 0 \\ 0 & , \text{ if } x = 0 \\ -1 & , \text{ else.} \end{cases}$$

This rule naturally follows from the distributivity of \odot w.r.t. \oplus . Indeed,

$$0 = (x \oplus (-x)) \odot y = (x \odot y) \oplus ((-x) \odot y),$$

which entails $(-x) \odot y = -x \odot y$.

We propose the following definition.

Definition 6 *Given a t-norm T , we define a symmetric pseudo-multiplication \odot as a binary operation on $[-1, 1]$:*

$$\textbf{(R5)} \text{ For } x, y \geq 0: x \odot y = T(x, y).$$

$$\textbf{(R6)} \text{ For other cases, } x \odot y = \text{sign}(x \cdot y) T(|x|, |y|).$$

Clearly, **R5** arises if we additionally require the monotonicity of \odot . If the construction of \oplus and \odot can be done as described above, and if in addition \odot is distributive w.r.t. \oplus , then $([-1, 1], \oplus, \odot)$ is a ring.

4. Results

We now examine the possibility of building symmetric pseudo-additions and multiplications leading to a ring. We consider the following four cases, covering all continuous t-conorms:

1. S is a strict t-conorm, with additive generator s ;
2. S is a nilpotent t-conorm, with additive generator s ;
3. S is the maximum operator;
4. S is an ordinal sum of continuous Archimedean t-conorms (i.e. combination of the above).

4.1. Strict t-conorms

Consider the symmetric pseudo-addition \oplus defined from a strict t-conorm S with additive generator s . Let us rescale \oplus to a binary operator U on $[0, 1]$ in the following way:

$$x \oplus y = 2U\left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y + \frac{1}{2}\right) - 1, \quad (11)$$

and we put $z := \frac{1}{2}x + \frac{1}{2}$, $t := \frac{1}{2}y + \frac{1}{2}$ (coordinates in $[0, 1]^2$).

Let us introduce $g : [-1, 1] \rightarrow [-\infty, \infty]$ as follows:

$$g(x) := s(x) \text{ for positive } x, \quad g(x) := -s(-x) \text{ for negative } x, \quad (12)$$

i.e. g is a symmetrization of s , and is clearly a strictly increasing function. We introduce also another function $u : [0, 1] \rightarrow [-\infty, \infty]$ defined by $u(x) = g(2x - 1)$ that is strictly increasing and satisfies $u(\frac{1}{2}) = 0$.

The following can be shown (originally established in¹⁰).

Proposition 2 *Let S be a strict t-conorm with additive generator s , \oplus the corresponding pseudo-addition, and g, u defined as above. Then:*

- (i) $x \ominus y = g^{-1}(g(x) - g(y))$ for any $x, y \in [0, 1]$;
- (ii) $x \oplus y = g^{-1}(g(x) + g(y))$ for any $x, y \in [-1, 1]$;
- (iii) $U(z, t) = u^{-1}(u(z) + u(t))$ for any $z, t \in [0, 1]$.

Proof: (i) Follows immediately from the definition of g and \ominus .

(ii) Clearly $x \oplus y = s^{-1}(s(x) + s(y)) = g^{-1}(g(x) + g(y))$ when $x, y \in [0, 1]$. When x, y are negative, it holds due to **R2** that

$$\begin{aligned} x \oplus y &= -g^{-1}(g(|x|) + g(|y|)) \\ &= g^{-1}(-(g(|x|) + g(|y|))) \\ &= g^{-1}(g(x) - g(y)). \end{aligned}$$

When x is positive and y is negative, we write $x \oplus y = x \ominus (-y)$ and use (i). The result then follows from the symmetry of g .

(iii) Using (11), the definition of z, t and (ii), we get

$$\begin{aligned} 2U(z, t) - 1 &= g^{-1}(g(x) + g(y)) \\ &= g^{-1}(g(2z - 1) + g(2t - 1)) \\ &= g^{-1}(u(z) + u(t)). \end{aligned}$$

Now observe that for any x and y , the equality $g^{-1}(y) = x$ is equivalent to $y = g(x) = u(\frac{1}{2} + \frac{1}{2}x)$, which in turn is equivalent to $u^{-1}(y) = \frac{1}{2} + \frac{1}{2}x = \frac{1}{2} + \frac{1}{2}g^{-1}(y)$. This shows that

$$U(z, t) = \frac{1}{2}g^{-1}(u(z) + u(t)) + \frac{1}{2} = u^{-1}(u(z) + u(t)).$$

□

Corollary 1 *Under the assumptions of Proposition 2, U is a uninorm that is continuous (except in $(0, 1)$ and $(1, 0)$), is strictly increasing on $]0, 1[^2$ and has neutral element $\frac{1}{2}$. Moreover, the t -norm T_U induced is the dual of S .*

Proof: Since u is strictly increasing and $u(\frac{1}{2}) = 0$, U is a uninorm for which statement (ii) of Proposition 1 holds. Now, using (6) and (7) with $e = \frac{1}{2}$, we obtain $t(x) = -s(x - 1)$, so that T_U and S are dual. □

Now we can state the main result.

Theorem 1 *Let S be a strict t -conorm with additive generator s and \oplus the corresponding pseudo-addition. Then it holds that:*

- (i) $(]-1, 1[, \oplus)$ is an Abelian group.
- (ii) There exists no pseudo-multiplication such that $(]-1, 1[, \oplus, \odot)$ is a ring.

Proof: (i) Since \oplus is a uninorm with neutral element $\frac{1}{2}$ transposed in $[-1, 1]^2$, commutativity and associativity hold, and the neutral element is 0. Moreover, symmetry holds since for any $x \in]-1, 1[$ we can show that

$$\begin{aligned} x \oplus (-x) &= 2U\left(\frac{1}{2}x + \frac{1}{2}, -\frac{1}{2}x + \frac{1}{2}\right) - 1 \\ &= 2u^{-1}(u(z) + u(1 - z)) = 2u^{-1}(u(z) - u(z)) - 1 = 1 \end{aligned}$$

using $u(z) = -u(1 - z)$, for all $z \in [0, 1]$.

(ii) By definition of the pseudo-multiplication, it is associative with neutral element 1, so only distributivity w.r.t. \oplus remains to be verified. If T is the t -norm associated to \oplus , T is distributive w.r.t. S if and only if $S = \max^{11}$. Since the maximum operator is not a strict t -conorm, the construction cannot work. □

4.2. Nilpotent t -conorms

For a nilpotent t -conorm with additive generator s , it is easy to see that statement (i) in Proposition 2 is still valid. However, the construction does not lead to an associative operator, as it is easily seen in the following example. Consider the Łukasiewicz t -conorm $S_L(x, y) = (x + y) \wedge 1$ with additive generator $s_L(x) = x$. Then $x \ominus y = x - y$. Let $x = -0.3$, $y = 0.6$, and $z = 0.6$, then we obtain:

$$\begin{aligned} x \oplus (y \oplus z) &= -0.3 \oplus (0.6 \oplus 0.6) = -0.3 \oplus 1 = 0.7 \\ (x \oplus y) \oplus z &= (-0.3 \oplus 0.6) \oplus 0.6 = 0.3 \oplus 0.6 = 0.9. \end{aligned}$$

Hence, we can even not build a group in this case.

4.3. The maximum operator

The case of maximum has already been studied by Grabisch in⁵, in a general setting where the underlying universe is a symmetric totally ordered set, i.e. a structure $L = L^+ \cup L^-$, where L^+ is a complete linear lattice with top $\mathbb{1}$ and bottom $\mathbb{0}$, $L^- := \{-x \mid x \in L^+\}$, and $-x \leq -y$ if and only if $x \geq y$, with the convention $-\mathbb{0} = \mathbb{0}$. We are interested in $L^+ = [0, 1]$, although subsequent results are valid in the general case. The symmetric maximum and minimum, denoted \mathbb{V} and \mathbb{W} , are defined as follows.

$$a \mathbb{V} b := \begin{cases} -(|a| \vee |b|) & , \text{ if } b \neq -a \text{ and } |a| \vee |b| \in \{-a, -b\} \\ \mathbb{0} & , \text{ if } b = -a \\ |a| \vee |b| & , \text{ else} \end{cases} \quad (13)$$

$$a \mathbb{W} b := \begin{cases} -(|a| \wedge |b|) & , \text{ if } \text{sign } a \neq \text{sign } b \\ |a| \wedge |b| & , \text{ else.} \end{cases} \quad (14)$$

The structure $(]-1, 1[, \mathbb{V})$ fails to be a group since \mathbb{V} is not always associative. For example,

$$(0.5 \mathbb{V} 0.8) \mathbb{V} (-0.8) = 0.8 \mathbb{V} (-0.8) = 0,$$

which differs from

$$0.5 \mathbb{V} (0.8 \mathbb{V} (-0.8)) = 0.5 \mathbb{V} 0 = 0.5.$$

A careful study in⁷ has led to the following result. Let us rephrase somewhat differently our requirements for symmetric maximum and minimum:

- (C1) \mathbb{V} and \mathbb{W} coincide with \vee and \wedge on L^+ , respectively;
- (C2) \mathbb{V} and \mathbb{W} are associative and commutative on L ;
- (C3) $-a$ is the symmetric element of a , i.e. $a \mathbb{V} (-a) = \mathbb{0}$, for $a \notin \{\mathbb{1}, -\mathbb{1}\}$;
- (C4) $-(a \mathbb{V} b) = (-a) \mathbb{V} (-b)$ and $-(a \mathbb{W} b) = (-a) \mathbb{W} b$, for any $a, b \in L$.

Condition C1 corresponds to R1 and R5, while conditions C3 and C4 correspond to R2, R3 and R4. Condition C2 expresses group properties. The following result shows that this task is impossible⁷.

Proposition 3 *We consider conditions (C1), (C3), (C4), and denote by (C4+) condition (C4) when a and b are restricted to L^+ . Then:*

1. *Conditions (C1) and (C3) imply that associativity cannot hold for \oplus .*
2. *Under conditions (C1) and (C4+), \oplus is neutral for \oplus . If we require in addition associativity, then $|a \oplus (-a)| \geq |a|$. Further, if we require isotonicity of \oplus , then $|a \oplus (-a)| = |a|$.*
3. *Under conditions (C1), (C2) and (C3), no operation is associative on a larger domain than \oplus defined by (13).*
4. *\oplus is associative for any expression involving a_1, \dots, a_n , $a_i \in L$, such that $\bigvee_{i=1}^n a_i \neq -\bigwedge_{i=1}^n a_i$.*
5. *The unique \oplus satisfying (C1) and (C4) is given by (14), and is associative. \oplus is distributive w.r.t. \oplus on L^+ and L^- separately.*

The conflict between associativity (C2) and symmetry (C3) is not surprising if we realize that for $S = \vee$, any element in $]0, 1]$ can be absorbing, if combined with smaller elements. Hence the basic observation stated in the beginning of Section 3 applies.

Rescaling the symmetric maximum to a binary operator U_{\max} on $[0, 1]$ leads to:

$$U_{\max}(x, y) := \begin{cases} \min(x, y) & , \text{ if } x + y < 1 \\ \frac{1}{2} & , \text{ if } x + y = 1 \\ \max(x, y) & , \text{ if } x + y > 1. \end{cases}$$

As it lacks associativity, this operator is not a uninorm, but it is still very much related to it. Indeed, the operator U_{\max} coincides (except on the antidiagonal $x + y = 1$) with two typical idempotent uninorms (either taking min or max on this antidiagonal)². However, it can also be obtained by a limit procedure starting from a representable uninorm U (by iteratively modifying its additive generator u)¹³.

4.4. Ordinal sums

Let us start with a simple case where S is defined by a strict t-conorm S_1 in the upper corner $[a, 1]^2$ for some $0 < a < 1$, and coincides with maximum elsewhere, that is:

$$S(x, y) = \begin{cases} (1 - a) S_1\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) + a & , \text{ if } x, y \in [a, 1] \\ x \vee y & , \text{ else.} \end{cases} \quad (15)$$

Following our construction, we define $x \oplus y = S(x, y)$ on $[0, 1]^2$ (this is **R1**). Due to **R2**, **R3** and **R4**, it remains to define $x \ominus_S y$, for $x, y \in [0, 1]$, $x \geq y$. The following can be shown.

Proposition 4 Let S be defined as in (15), and s_1 be an additive generator of S_1 . Then it holds that:

(i) For $(x, y) \in [a, 1]^2$, $x \geq y$:

$$x \ominus_S y = \begin{cases} (1-a)s_1^{-1} \left(s_1 \left(\frac{x-a}{1-a} \right) - s_1 \left(\frac{y-a}{1-a} \right) \right) + a & , \text{ if } x > y \\ 0 & , \text{ else;} \end{cases} \quad (16)$$

(ii) For $(x, y) \in [0, 1]^2 \setminus [a, 1]^2$, $x \geq y$:

$$x \ominus_S y = x \ominus'_V y. \quad (17)$$

Proof: (i) By definition,

$$x \ominus_S y = \inf \left\{ z \geq a \mid S \left(\frac{y-a}{1-a}, \frac{z-a}{1-a} \right) \geq \frac{x-a}{1-a} \right\} \wedge \inf \{ z < a \mid y \vee z \geq x \}.$$

The second term does not exist unless $x = y$, and in this case it is 0. Since s_1 is strictly increasing, the inequality in the first term is equivalent to

$$\frac{z-a}{1-a} \geq s_1^{-1} \left(s_1 \left(\frac{x-a}{1-a} \right) - s_1 \left(\frac{y-a}{1-a} \right) \right),$$

and the result follows.

(ii) Observe that necessarily $y \leq a$. Then we have

$$x \ominus_S y = \inf \{ z \mid y \vee z \geq x \} = x \ominus'_V y.$$

□

Note that \oplus is discontinuous on the line $x = y$, even if $x \geq a$. Let us now express \oplus using a generator function, where possible.

Proposition 5 Let us define the generator function $g : [-1, -a[\cup \{0\} \cup]a, 1] \rightarrow [-\infty, \infty]$ by:

$$g(x) = \begin{cases} s_1 \left(\frac{x-a}{1-a} \right), & \text{if } x \in [a, 1] \\ -g(-x), & \text{if } x \in [-1, -a[\\ 0, & \text{if } x = 0 \end{cases}$$

with the convention $g^{-1}(0) = 0$. For any x, y such that $(|x|, |y|) \in [a, 1]^2$, we have:

$$x \oplus y = g^{-1}(g(x) + g(y)).$$

Proof: For $y > 0$, it holds that $g^{-1}(y) = x$ is equivalent to $y = g(x) = s_1 \left(\frac{x-a}{1-a} \right)$, which in turn is equivalent to $s_1^{-1}(y) = \frac{x-a}{1-a} = \frac{g^{-1}(y)-a}{1-a}$. Hence, we have $g^{-1}(y) = s_1^{-1}(y)(1-a) + a$. Substituting in (15) and (16) and considering symmetrization lead to the desired result for $x \neq -y$. In case $x = -y$, we have $x \oplus y = g^{-1}(0) = 0$. □

Let us study the associativity of \oplus . Clearly, since an additive generator exists, \oplus is associative when the absolute value of the arguments belong to $[a, 1]$. Let us consider the problematic case $(b \oplus c) \ominus_S c$ when $0 < b < a < c < 1$. We have $(b \oplus c) \ominus_S c = (b \vee c) \ominus_S c = c \ominus_S c = 0$. However, $b \oplus (c \ominus_S c) = b$. Hence associativity does not hold everywhere.

It is easily verified that putting a second strict t-conorm S_2 on $[0, a]^2$ in order to cover the diagonal does not change the problem. Let us consider as above $0 < b < a < c < 1$. Then $(b \oplus c) \ominus_S c = (b \vee c) \ominus_S c = 0$ again, and associativity is not fulfilled. We conclude that ordinal sums cannot lead to associative operators, although they result in “more associative” operators than maximum does.

5. On the relation with ordered Abelian groups

The result on strict t-conorms could have been shown using results known in the theory of ordered groups. We introduce the necessary definitions (see e.g.^{9,4}).

Definition 7 *Let (W, \leq) be a linearly ordered set, having top and bottom denoted \top and \perp , a particular nonextremal element e , and let us consider an internal binary operation \oplus on W , and a unary operation \ominus on W such that $x \leq y$ if and only if $\ominus(x) \geq \ominus(y)$:*

- (i) *$(W, \leq, \oplus, \ominus, e)$ is an ordered Abelian group (OAG) if it satisfies for all nonextremal elements x, y, z :*
 - (i1) $x \oplus y = y \oplus x$;
 - (i2) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
 - (i3) $x \oplus e = x$;
 - (i4) $x \oplus (\ominus(x)) = e$;
 - (i5) $x \leq y$ implies $x \oplus z \leq y \oplus z$.
- (ii) *$(W, \leq, \oplus, \ominus, e)$ is an extended ordered Abelian group (OAG⁺) if in addition*
 - 1. $\top \oplus x = \top$, $\perp \oplus x = \perp$ for all x , $\ominus(\top) = \perp$, $\ominus(\perp) = \top$;
 - 2. if x and y are non-extremal, then $x \oplus y$ is non-extremal.

Clearly, our concern is to find an OAG⁺, with $W = [-1, 1]$, $\top = 1$, $\perp = -1$, $\ominus = -$, and \oplus corresponds to our operation \oplus .

Definition 8 (i) *An isomorphism ϕ of an OAG (resp. OAG⁺) $\mathbf{W} = (W, \leq, \oplus, \ominus, e)$ onto an OAG (resp. OAG⁺) $\mathbf{W}' = (W', \leq', \oplus', \ominus', e')$ is a one-to-one mapping from W onto W' preserving the structure, i.e. such that*

- (i1) $\phi(x \oplus y) = \phi(x) \oplus' \phi(y)$;
- (i2) $\phi(\ominus x) = \ominus' \phi(x)$;
- (i3) $\phi(e) = e'$;
- (i4) $x \leq y$ if and only if $\phi(x) \leq' \phi(y)$.

- (ii) \mathbf{W} is a substructure of \mathbf{W}' if $W \subset W'$ and the structure of \mathbf{W} is the restriction of the structure of \mathbf{W}' to W , i.e. $x \oplus y = x \oplus' y$, $\ominus x = \ominus' x$, $e = e'$, and $x \leq y$ if and only if $x \leq' y$, for all $x, y \in W$.
- (iii) An isomorphic embedding of \mathbf{W} into \mathbf{W}' is an isomorphism of \mathbf{W} onto a substructure of \mathbf{W}' .

Observe that in (i) above, in the case of OAG^+ , we have in addition $\phi(\top) = \top'$ (due to (i1) with $x = \top$), and $\phi(\perp) = \perp'$ (due to (i2)).

Definition 9 (i) An OAG \mathbf{W} is dense if there is no least positive element, i.e. an element $x \in W$ such that $x > e$, and there is no $y \in W$ such that $e < y < x$.

- (ii) An OAG \mathbf{W} is completely ordered if each non-empty bounded $X \subset W$ has a least upper bound.

Obviously, $(]-1, 1[, \leq, \oplus, -, 0)$ is dense and completely ordered, the same holds for the closed interval.

Theorem 2 If \mathbf{W} is a completely ordered and dense OAG, then it is isomorphic to $(\mathbb{R}, \leq, +, -, 0)$.

The same result holds if \mathbf{W} is an OAG^+ and if \mathbb{R} is replaced by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. This shows that \oplus has necessarily the following form:

$$x \oplus y = \phi^{-1}[\phi(x) + \phi(y)], \quad (18)$$

where $\phi : [-1, 1] \rightarrow \overline{\mathbb{R}}$ is one-to-one, odd, increasing, and satisfies $\phi(0) = 0$. Clearly, ϕ corresponds to the function g of Proposition 2.

6. Concluding remarks

In this paper we have studied whether it is possible to introduce a ring structure on $]-1, 1[$, using a t-conorm and a t-norm. The results show that the answer is negative. In the particular case of the maximum and minimum, the symmetric maximum and minimum⁵ are the best possible choices: they provide a structure that is not a ring, but is the closest possible to a ring in a well-defined sense.

One may be interested in finding a ring structure on finite ordinal scales. However, it is well known that there exist no strict t-conorms on finite ordinal scales¹². Hence, there is no chance to obtain an Abelian group. Nilpotent t-conorms lead to disadvantageous results as we have shown above. Hence, one can conclude that the best choice seems to be the symmetric maximum.

Finally, we remark that this structure together with the symmetric minimum is the closest possible to a ring, due to the fact that the only chance to get distributivity is with $S = \max$.

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References

1. B. De Baets. Coimplicators, the forgotten connectives. *Tatra Mt. Math. Publ.*, 12:229–240, 1997.
2. B. De Baets. Idempotent uninorms. *European J. Oper. Res.*, 118:631–642, 1999.
3. J. Fodor, B. De Baets, and T. Calvo. Characterization of uninorms with given continuous underlying t-norms and t-conorms. submitted.
4. L. Fuchs. *Partially Ordered Algebraic Systems*. Addison-Wesley, 1963.
5. M. Grabisch. Symmetric and asymmetric fuzzy integrals: the ordinal case. In *6th Int. Conf. on Soft Computing (Iizuka 2000)*, Iizuka, Japan, October 2000.
6. M. Grabisch. The symmetric Sugeno integral. *Fuzzy Sets and Systems*, 139:473–490.
7. M. Grabisch. The Möbius function on symmetric ordered structures and its application to capacities on finite sets. *Discrete Mathematics*, submitted.
8. G. Grätzer. *General Lattice Theory*. Birkhäuser, 2nd edition, 1998.
9. P. Hájek, T. Havránek, and R. Jiroušek. *Uncertain Information Processing in Expert Systems*. CRC Press, 1992.
10. E.P. Klement, R. Mesiar, and E. Pap. On the relationship of associative compensatory operators to triangular norms and conorms. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, 4:129–144, 1996.
11. E.P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*. Kluwer Academic Publishers, 2000.
12. G. Mayor and J. Torrens. On a class of operators for expert systems. *Int. J. of Intelligent Systems*, 8:771–778, 1993.
13. R. Mesiar and M. Komorníková. Triangular norm-based aggregation of evidence under fuzziness. In: B. Bouchon-Meunier, editor, *Aggregation and Fusion of Imperfect Information*, Studies in Fuzziness and Soft Computing, pages 11–35. Physica Verlag, 1998.
14. T. Murofushi and M. Sugeno. Fuzzy t-conorm integrals with respect to fuzzy measures : generalization of Sugeno integral and Choquet integral. *Fuzzy Sets and Systems*, 42:57–71, 1991.
15. A. Tversky and D. Kahneman. Advances in prospect theory: cumulative representation of uncertainty. *J. of Risk and Uncertainty*, 5:297–323, 1992.
16. S. Weber. \perp -decomposable measures and integrals for archimedean t-conorms \perp . *J. Math. Anal. Appl.*, 101:114–138, 1984.
17. R. Yager and A. Rybalov. Uninorm aggregation operators. *Fuzzy Sets and Systems*, 80:111–120, 1996.